Solution. Suppose an ellipse and a hyperbola with foci F_1 and F_2 intersect at P. Then their tangents at that point will be the bisectors of the exterior and interior angles F_1PF_2, respectively. Therefore they are perpendicular.

Theorem 1.2. *Suppose the chord PQ contains a focus F_1 of the ellipse and R is the intersection of the tangents to the ellipse at P and Q. Then R is the center of an excircle of the triangle F_2PQ, and F_1 is the tangency point of that circle and the side PQ* (Figure 1.14).

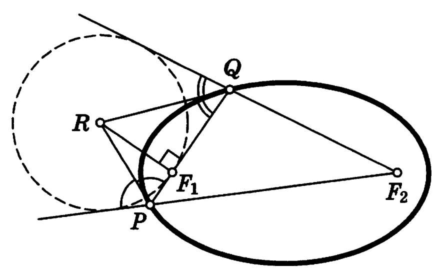

FIGURE 1.14

Proof. By the optical property, PR and QR are the bisectors of the exterior angles of the triangle F_2PQ. Therefore R is the center of an excircle. The tangency point (call it F_1') of the excircle and the corresponding side and the point F_2 cut the perimeter of the triangle into equal parts, i.e., $F_1'P+PF_2 = F_2Q+QF_1'$. But F_1 has this property and there is only one such point. Hence F_1' and F_1 coincide. ☐

Corollary. *The straight line connecting a focus of an ellipse and the intersection of the tangents to the ellipse at the ends of a chord containing that focus is perpendicular to the chord.*

For the hyperbola, Theorem 1.2 is also true but the excircle should be replaced by the incircle.

1.4. The isogonal property of conics

The optical property yields elementary proofs of some amazing results.

Theorem 1.3. *From any point P outside an ellipse draw two tangents to the ellipse, with tangency points X and Y. Then the angles F_1PX and F_2PY are equal (F_1 and F_2 are the foci of the ellipse).*

Proof. Let F_1', F_2' be the reflections of F_1 and F_2 in PX and PY, respectively (Figure 1.15).

Then $PF_1' = PF_1$ and $PF_2' = PF_2$. Moreover, the points F_1, Y and F_2' lie on a line (because of the optical property). The same is true for the points F_2, X and F_1'. Thus $F_2F_1' = F_2X + XF_1 = F_2Y + YF_1 = F_2'F_1$.

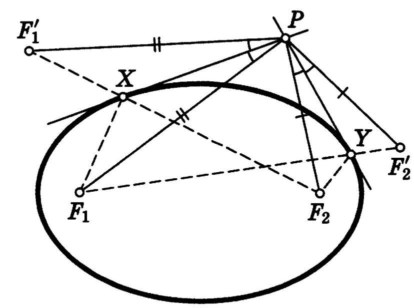

FIGURE 1.15

Thus, the triangles PF_2F_1' and PF_1F_2' are equal (having three equal sides). Therefore

$$\angle F_2PF_1 + 2\angle F_1PX = \angle F_2PF_1' = \angle F_1PF_2' = \angle F_1PF_2 + 2\angle F_2PY.$$

Hence $\angle F_1PX = \angle F_2PY$, which is the desired result.[1] □

Figure 1.16 shows that a similar property holds for the hyperbola.[2]

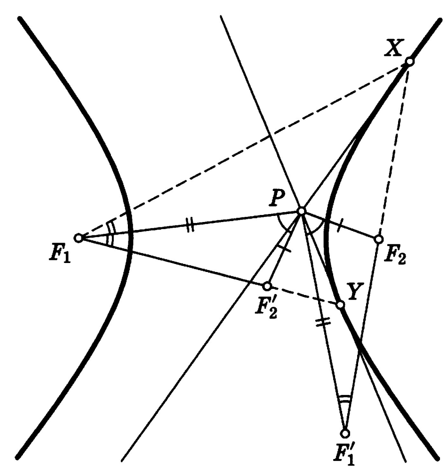

FIGURE 1.16

Suppose now that the ellipse (or hyperbola) with foci F_1 and F_2 is inscribed in triangle ABC. It follows from the above that $\angle BAF_1 = \angle CAF_2$, $\angle ABF_1 = \angle CBF_2$ and $\angle ACF_1 = \angle BCF_2$.

We shall show in 2.3 that, in a plane, for any (with rare exceptions) point X there is a unique point Y such that X and Y are the foci of a

[1]We consider the case when F_1 and F_2 are inside the angle $F_1'PF_2'$ and F_1 lies inside the angle F_2PF_1'. In the remaining cases the arguments are similar.

[2]The reader should check two cases: when the tangency points are either on different branches or on the same branch.

conic tangent to each side of a triangle. Such Y is said to be the *isogonal conjugate of* X with respect to the triangle.

The construction used in the proof of Theorem 1.3, allows one to obtain yet another interesting result. Since the triangles PF_2F_1' and $PF_2'F_1$ are equal, the angles $PF_1'F_2$ and PF_1F_2' are also equal. Therefore

$$\angle PF_1X = \angle PF_1'F_2 = \angle PF_1F_2' = \angle PF_1Y.$$

Thus we have proved the following generalization of Theorem 1.2.

Theorem 1.4. *In the notation of Theorem* 1.3, *the line* F_1P *is the bisector of the angle* XF_1Y (Figure 1.17).

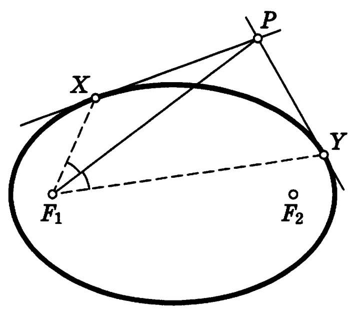

FIGURE 1.17

Theorem 1.5. *The locus of points from which a given ellipse is seen at a right angle* (*i.e., the tangents to the ellipse drawn from such a point are perpendicular*) *is a circle centered at the center of the ellipse* (Figure 1.18).

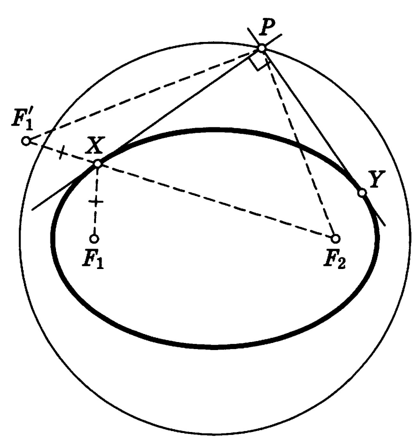

FIGURE 1.18

Proof. Let F_1 and F_2 be the foci of the ellipse and suppose that the tangents to the ellipse at X and Y intersect in P. Reflecting F_1 in PX we have a point F_1'. It follows from Theorem 1.3 that $\angle XPY = \angle F_1'PF_2$ and $F_1'F_2 = F_1X + F_2X$, i.e., the length of the segment $F_1'F_2$ equals the major axis of the ellipse (the length of the rope tying the goat). The angle $F_1'PF_2$ is right if and only if $F_1'P^2 + F_2P^2 = F_1'F_2^2$ (by the Pythagorean theorem). Therefore XPY is a right angle if and only if $F_1P^2 + F_2P^2$ equals the square of the major axis of the ellipse. But it is not difficult to see that this condition defines a circle. Indeed, suppose F_1 has Cartesian coordinates (x_1, y_1), and F_2 has coordinates (x_2, y_2). Then the coordinates of the desired points P satisfy the condition

$$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 = C,$$

where C is the square of the major axis. But since the coefficients of x^2 and y^2 are equal (to 2) and the coefficient of xy is zero, the set of points satisfying this condition is a circle. By virtue of symmetry, its center is the midpoint of the segment F_1F_2. □

For the hyperbola such a circle does not always exist. When the angle between the asymptotes of the hyperbola is acute, the radius of the circle is imaginary. If the asymptotes are perpendicular, then the circle degenerates into the point which is the center of the hyperbola.

Example. Given points $P_1, \ldots, P_n$ and numbers $k_1, \ldots, k_n$ and C, the locus of points X such that $k_1XP_1^2 + \cdots + k_nXP_n^2 = C$ is a circle, known as the *Fermat–Apollonius circle.* Clearly, it may have an imaginary radius (when?).

Theorem 1.6. *Suppose a string is put on an ellipse α and then pulled tight using a pencil. If the pencil is rotated about the ellipse, it will traverse another ellipse confocal with α* (Figure 1.19).

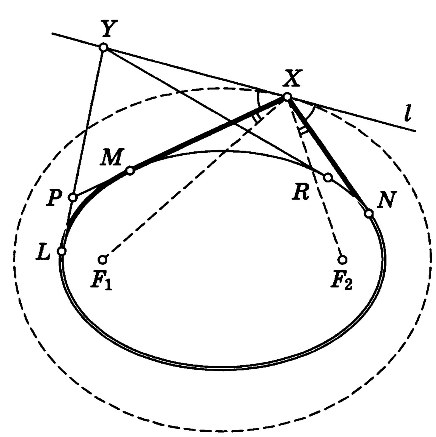

FIGURE 1.19

Proof. Clearly, the new figure (call it α_1) has a smooth boundary. We shall show that at each point X on α_1 the tangent to the new curve coincides with the bisector of the exterior angle F_1XF_2.

Let XM and XN be the tangents to α. Then $\angle F_1XN = \angle F_2XM$, and hence the bisector l of the exterior angle NXM coincides with the bisector of the exterior angle F_1XF_2. Call it l.

Let Y be an arbitrary point on l and YL and YR the tangents to α, as shown in Figure 1.19. We assume that Y lies "to the left" of X; the other case is argued similarly.

Let P be the intersection of the lines XM and YL. It is easy to see that $YN < YR + \smile RN$, and $\smile LM < LP + PM$. Moreover, since l is the exterior bisector of the angle NXP, we have $PX + XN < PY + YN$. Therefore

$$\begin{aligned} MX + XN + \smile NM &< MX + XN + \smile NL + LP + PM \\ &= PX + XN + \smile NL + LP < PY + YN + \smile NL + LP \\ &= LY + YN + \smile NL \\ &< LY + YR + \smile RN + \smile NL = LY + YR + \smile RL \end{aligned}$$

(here the arcs are meant to be the arcs under the string). Therefore Y lies outside α_1. The same is true for any point Y on l. It follows that α_1 contains a single point of l, i.e., the line is tangent. It also follows at once that the obtained curve is convex.

Thus the sum of the distances to the foci F_1 and F_2 does not change with time. Therefore the trajectory of the pencil is an ellipse.

Here is a more rigorous approach to the last claim. Suppose X is outside the ellipse. Put the pencil at X and pull the string around it and around the ellipse. Let $f(X)$ be the length of the string and $g(X) = F_1X + F_2X$ (a point is understood as a pair of its coordinates; thus both f and g depend on a pair of real numbers). One can show that those functions are continuously differentiable and that the vectors $\operatorname{grad} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ and $\operatorname{grad} g = (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y})$ are nonzero at each point. Then, by the implicit function theorem, the curve traversed by the pencil with a string of fixed length (i.e., a level curve of f) is smooth (continuously differentiable). It now follows that the curve can be parametrized by a differentiable function $R = R(t)$ (this is again a pair of coordinate functions $x = x(t)$, $y = y(t)$) whose tangent vector is different from zero. As shown before, the tangent vector $\frac{dR}{dt} = (\frac{dx}{dt}, \frac{dy}{dt})$ of the curve is tangent to a level curve of g, i.e., it is perpendicular to $\operatorname{grad} g(R)$ at $R = R(t)$. Consider the function $g(R(t))$. Its derivative is

$$\frac{dg(R(t))}{dt} = \frac{\partial g}{\partial x}\frac{dx(t)}{dt} + \frac{\partial g}{\partial y}\frac{dy(t)}{dt} \equiv 0$$

(this is the orthogonality condition mentioned above), i.e., $g(R(t))$ is constant. This means that our curve lies on an ellipse with the same foci. Since any ray starting at F_1 must contain a point on our curve, the curve coincides with the ellipse. □

Problem 2. A $2n$-gon is circumscribed about a conic with focus F. Its sides are colored in black and white in an alternating pattern. Prove that the sum of the angles at which the black sides are seen from F equals 180°.

Problem 3. An ellipse is inscribed in a convex quadrilateral such that its foci lie on the (distinct) diagonals of the quadrilateral. Prove that the products of the opposite sides are equal.

1.5. Curves of second degree as projections of the circle

Given a circle, draw the perpendicular through its center to the plane of the circle and pick a point S on it. The lines connecting S to the points of the circle form a cone. Consider the section of the cone by a plane π intersecting all of its rulings and not perpendicular to its axis of symmetry.

Now inscribe in the cone two spheres touching π at points F_1 and F_2 (Figure 1.20).

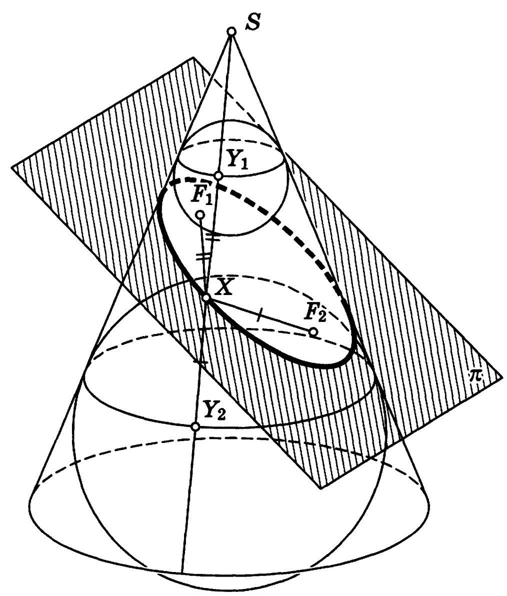

FIGURE 1.20

Let X be an arbitrary point on the intersection of the cone and the plane π. The ruling SX intersects the inscribed spheres at points Y_1 and Y_2. We have $XF_1 = XY_1$ and $XF_2 = XY_2$, since the segments of tangents to a sphere drawn from the same point are equal. Therefore $XF_1 + XF_2 = Y_1Y_2$. But Y_1Y_2 is the segment of the ruling lying between the two planes perpendicular to the axis of the cone, and its length does not depend on the choice of X. Hence the intersection of the cone with π is an ellipse. The ratio of its semiaxes depends on the tilt of the plane and, obviously, can take on any value. Therefore any ellipse can be obtained as a central projection of the circle.

A similar proof shows that if the secant plane is parallel to two rulings of the cone, then the cross-section is a hyperbola (Figure 1.21).